Published by Institute of Physics Publishing for SISSA
Received: *May 5, 2006*
Accepted: *June 15, 2006*
Published: *July 13, 2006*

On the AdS higher spin / O(N) vector model correspondence: degeneracy of the holographic image

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Abstract: We explore the conjectured duality between the critical O(N) vector model and minimal bosonic massless higher spin (HS) theory in AdS. In the free boundary theory, the conformal partial wave expansion (CPWE) of the four-point function of the scalar singlet bilinear is reorganized to make it explicitly crossing-symmetric and closed in the singlet sector, dual to the bulk HS gauge fields. We are able to analytically establish the factorized form of the fusion coefficients as well as the two-point function coefficient of the HS currents. We insist in directly computing the free correlators from bulk graphs with the unconventional branch. The three-point function of the scalar bilinear turns out to be an "extremal" one at $d = 3$. The four-point bulk exchange graph can be precisely related to the CPWs of the boundary dual scalar and its shadow. The flow in the IR by Legendre transforming at leading 1/N, following the pattern of double-trace deformations, and the assumption of degeneracy of the hologram lead to the CPWE of the scalar four-point function at IR. Here we confirm some previous results, obtained from more involved computations of skeleton graphs, as well as extend some of them from $d = 3$ to generic dimension $2 < d < 4$.

Keywords: AdS-CFT Correspondence, 1/N Expansion, Field Theories in Lower Dimensions.

Contents

1. Introduction

Duality between strongly coupled SYM in the large N limit and weakly coupled SUGRA is one of the forms of Maldacena's conjecture. Many interesting results have been obtained and many tests have been performed in this regime. However, much less is known about the bulk dual to perturbative gauge theories or even to free theory. It has been conjectured that the dual bulk theory to large N free gauge theory is a HS theory of Fradkin-Vasiliev type [1, 2]. A simpler scenario for testing these ideas has been proposed by Klebanov and Polyakov [3], concerning the bulk dual of the critical O(N) vector model. Vector models have always been useful in understanding features that arise in the more complicated case of gauge theories. Here one can use the vast experience in large-N limit of O(N) vector models to reconstruct the bulk theory. An analogous attempt has been started by Gopakumar [4] for the singlet bilinear sector (twist-two operators in $d = 4$) of the gauge theory; but despite the initial success in casting two and three-point function of scalar bilinears into AdS amplitudes, four-point correlators have remained a challenge.[1] The technical difficulty has been that one should include the whole tower of HS fields, dual to the HS conserved currents of the CFT at the boundary, in the exchange graphs since the OPE structure of the free field correlators involve the whole tower of conserved currents. A bulk theory consistently truncated to massless fields should be reflected somehow in a closure of the corresponding dual sector of CFT operators. We study the free four-point function of the scalar bilinear by means of a conformal partial wave expansion and reorganize it so as to involve only the minimal twist sector,[2] i.e. the conserved HS currents and their descendants, as required by the correspondence. The fusion coefficients are analytically checked to factorize in terms of two- and three-point function coefficients. A comparison with the correlators at the IR fixed point can be made by means of the amputation procedure that realizes the Legendre transformation connecting the two conformal theories at leading order in the large-N expansion, in close analogy with the effect of double-trace deformations in the gauge theory (see e.g. [6] and references therein). We pursue the view that both CFTs are on equal footing, related by Legendre transformation, and that one can compute directly the Witten graphs with either branch $\Delta_{+/-}$.[3] Our aim is to have an autonomous way to compute directly in the free theory, having in mind a possible extension to free gauge theories, with no need of Legendre transforming from a conjugate CFT that arises at leading large N but whose existence is otherwise uncertain. As a consequence, at $d = 3$ one has a vanishing three-point function for the scalar bilinear at IR. On the other hand, at UV the three-point function is nonzero due to the compensation of the vanishing coupling by a divergence of the corresponding Witten graph. This is similar to the case of extremal correlators, see e.g. [11]. The underlying assumption of a common bulk theory, degeneracy of the holographic image, is also consistent with the CPWE of the four-point correlators. Progress in the bulk side of the correspondence is considerably more difficult due to the complicated nature of the interacting HS theories on AdS. We use the CPWs to mimic the effect of the corresponding bulk exchange graphs, even though the CPW is generically only a part of the Witten graph and one can only hope that after including the whole tower of HS exchange the additional terms cancel out. In this direction, we study the scalar exchange in AdS and relate it to the corresponding CPW.

[1] See however [5] for recent progress and for a large set of references to proposals for string dual of large N gauge theories.

[2] Here we follow a suggestion in [2] for the free gauge theory case and turn it into a quantitative result.

[3] While preparing this text a paper by Hartman and Rastelli [7] appeared which also stresses this view.

The paper is organized as follows: we start by briefly describing the conjecture. Then we define the higher spin conserved currents and compute some of their correlators.[4] We then obtain the conformal partial waves of the HS currents to study the scalar bilinear four-point function, in an attempt to get closer to a bulk AdS formulation, and manipulate the results to achieve the closure of the bilinear sector. We then move to the IR critical theory by means of the amputation procedure and, based on the holographic degeneracy, predict the CPWE at IR including fusion coefficients. Finally, we explore at $d = 3$ the precise connection between the scalar exchange Witten graph and the corresponding scalar CPW of dimension $\Delta_- = 1$. Some useful formulas and details of the calculations are collected in various appendices.

2. The Klebanov-Polyakov conjecture

Let us briefly review the essentials of the conjecture. The singlet sector of the critical 3-dim O(N) vector model with the $(\vec{\varphi}^2)^2$ interaction is conjectured to be dual, in the large N limit, to the minimal bosonic theory in AdS_4 containing massless gauge fields of even spin. There is a one-to-one correspondence between the spectrum of currents and that of massless higher-spin fields. In addition we have a scalar bilinear J mapped to a bulk scalar ϕ. The AdS/CFT correspondence working in the standard way (conventional dimension Δ_+ for J) produces the correlation functions of the singlet currents in the interacting large N vector model at its IR critical point from the bulk action in AdS_4 by identifying the boundary term ϕ_0 of the field ϕ with a source in the dual field theory (cf. appendix F). At the same time, the correlators in the free theory are obtained by Legendre transforming the generating functional with respect to the source that couples to the scalar bilinear J; this corresponds on the AdS side to the procedure for extracting the correlation functions working with the unconventional branch Δ_- [3, 9]. However, we want to stress that one can directly compute the bulk graphs with the Δ_- branch, by using the boundary term A (cf. appendix F) as source in the boundary theory, and that the boundary correlator obtained is precisely related by Legendre transformation to the one computed with the standard Δ_+ branch.

3. Free O(N) vector model

We start by considering N elementary real fields φ^a in d-dimensional Minkowski space, vectors under the global O(N) symmetry and Lorentz scalars with canonical scaling dimension $\delta = d/2 - 1$(in what follows we switch to Euclidean space). They satisfy the free equation of motion $\partial^2 \varphi^a = 0$. We normalize their two-point function as

$$\langle \varphi^a(x_1)\varphi^b(x_2)\rangle = \frac{\delta^{ab}}{r_{12}^{\delta}}, \qquad a,b = 1,\ldots,N, \tag{3.1}$$

where $\quad r_{ij} = |x_i - x_j|^2 = |x_{ij}|^2$.

[4] In particular, the coefficient of the two-point function is analytically obtained. In the literature, this result is only known as extrapolation of an expression obtained by computer algebraic manipulations [8].

3.1 HS conserved currents

In this free theory there is an infinite tower of (even) higher spin currents, bilinear in the elementary fields, which are totally symmetric, traceless and conserved. These three properties fix their form, their precise expression can be found in [8, 10]. We will only need them in the following form (assuming normal order and omitting free indices)

$$J_l = \sum_{k=0}^{l} a_k\, \partial^k \overrightarrow{\varphi} \cdot \partial^{l-k} \overrightarrow{\varphi} \; - traces, \tag{3.2}$$

with[5]

$$a_k = a_{l-k} = \frac{1}{2}(-1)^k \binom{l}{k} \frac{(\delta)_l}{(\delta)_k (\delta)_{l-k}}. \tag{3.3}$$

Note that this convention means

$$J_l = \overrightarrow{\varphi} \cdot \partial^l \overrightarrow{\varphi} \; + \cdots , \tag{3.4}$$

where the ellipsis stands for terms involving derivatives of both fields. They are conformal quasi-primaries, minimal twist operators with scaling dimension

$$\Delta_l = d - 2 + l = 2\delta + l. \tag{3.5}$$

The AdS/CFT Correspondence relates them to massless HS bulk field since the canonical dimension Δ_l precisely saturates the unitarity bound for totally symmetric traceless rank l tensors ($l > 0$, even).

3.2 Two- and three-point functions

The singlet bilinear sector is completed by adding to the above list the scalar bilinear $J = \overrightarrow{\varphi}^2$, spin-zero current, with canonical dimension $\Delta_J = d - 2 = 2\delta$. At $d = 3$ its bulk partner is a conformally coupled scalar.

Let us compute the two-point function of the HS currents and the three-point function of two spin-zero and a HS current.

The conformal symmetry fixes the form of the two-point function up to a constant (A.3),

$$\langle J_{l\;\mu_1 \dots \mu_l}(x) J_{l\;\nu_1 \dots \nu_l}(0)\rangle = C_{J_l}\, r^{-2\delta - l}\; sym\{I_{\mu_1\nu_1}(x) \dots I_{\mu_l\nu_l}(x)\}. \tag{3.6}$$

To find the coefficient it is then sufficient to look at the term $2^l \frac{x \dots x}{r^l}$ involving x $2l$ times. By Wick contracting we get

$$\langle J_l(x) J_l(y)\rangle = \sum_{k,s=0}^{l} a_k\, a_s \{\partial_x^k \partial_y^s \langle \varphi^a(x)\varphi^b(y)\rangle \partial_x^{l-k} \partial_y^{l-s} \langle \varphi^a(x)\varphi^b(y)\rangle + (s \leftrightarrow l - s)\} - \text{traces}. \tag{3.7}$$

[5] The Pochhammer symbol $(q)_r = \frac{\Gamma(q+r)}{\Gamma(q)}$.

Using now the symmetry $a_k = a_{l-k}$, trading ∂_y by $-\partial_x$ and taking $y = 0$ we get, up to trace terms,

$$2N \sum_{k,s=0}^{l} a_k\, a_s\, (\partial^{k+s} r^{-\delta})\, (\partial^{2l-k-s} r^{-\delta}) = 2^{2l+1} N \frac{x \ldots x}{r^{2\delta+2l}} \sum_{k,s=0}^{l} a_k\, a_s\, (\delta)_{k+s}\, (\delta)_{2l-k-s}. \tag{3.8}$$

The double summation is done using generalized hypergeometric series in appendix B giving $\frac{1}{4} l!(2\delta - 1 + l)_l$. The coefficient of the two-point function is then $(l > 0)$

$$C_{J_l} = 2^{l-1} N l! (2\delta - 1 + l)_l \tag{3.9}$$

that coincides with the extrapolated result reported by Anselmi [8]. Analogously, the three-point function form is dictated by the conformal symmetry (see A.1)

$$\langle J(x_1) J(x_2) J_{l\, \nu_1 \ldots \nu_l}(x_3) \rangle = C_{JJJ_l}\, (r_{12}\, r_{23}\, r_{31})^{-\delta}\, \lambda^{x_3}_{\mu_1 \ldots \mu_l}(x_1, x_2). \tag{3.10}$$

Now we focus on the coefficient of $2^l \frac{x_{31} \ldots x_{31}}{r^l}$ involving x_{31} $2l$ times after Wick-contracting,

$$\begin{aligned} \langle J(x_1) J(x_2) J_l(x_3) \rangle &= \langle\!\langle \varphi^a(x_1) \varphi^b(x_2) \rangle\!\rangle \langle\!\langle \varphi^b(x_2) \varphi^c(x_3) \rangle\!\rangle \partial^l_{x_3} \langle\!\langle \varphi^c(x_3) \varphi^a(x_1) \rangle\!\rangle + \text{permutations} + \cdots \\ &= 4N (r_{12} r_{23})^{-\delta} 2^l (\delta)_l \frac{x_{31} \ldots x_{31}}{r_{31}^{\delta+l}} + \cdots . \end{aligned} \tag{3.11}$$

Finally, we get for the coefficient of the three-point function $(l > 0)$

$$C_{JJJ_l} = 2^{l+2} N (\delta)_l\,. \tag{3.12}$$

The corresponding values for the scalar are $C_J = 2N$ and $C_{\mathrm{JJJ}} = 8N$.

3.3 Scalar four-point function

The four-point function contains much more dynamical information encoded in a function of two conformal invariant cross-ratios which is not fixed by conformal symmetry. Still, its form is constrained by the OPE of any two fields and therefore the contributions of operators of arbitrary spin, including their derivative descendants, can be unveiled. The connected part of the spin-zero current four-point function is obtained by Wick contractions

$$\langle J(x_1) J(x_2) J(x_3) J(x_4) \rangle_{free,conn} = \frac{16N}{(r_{12} r_{34})^{2\delta}} \left\{ u^\delta + \left(\frac{u}{v}\right)^\delta + u^\delta \left(\frac{u}{v}\right)^\delta \right\}, \tag{3.13}$$

where $u = \frac{r_{12}\, r_{34}}{r_{13}\, r_{24}}$ and $v = \frac{r_{14}\, r_{23}}{r_{13}\, r_{24}}$. Diagrammatically, it is given by the three boxes in figure 1.

Following Klebanov and Polyakov [3] we notice that the leading term in the box diagram

$$\frac{1}{(r_{12} r_{23} r_{34} r_{41})^{1/2}} \sim \frac{1}{(r_{12} r_{34})^{1/2}} \frac{1}{r_{13}} \tag{3.14}$$

in the direct channel limit $1 \to 2, 3 \to 4$ correctly reproduces the contribution of the scalar J with dimension $\Delta = \Delta_J = 1$ to the double OPE (see, e.g. [11]), which in general reads

$$\langle J(x_1) J(x_2) J(x_3) J(x_4) \rangle \sim \frac{1}{(r_{12} r_{34})^{\Delta_J - \Delta/2}} \frac{1}{(r_{13})^{\Delta}}. \tag{3.15}$$

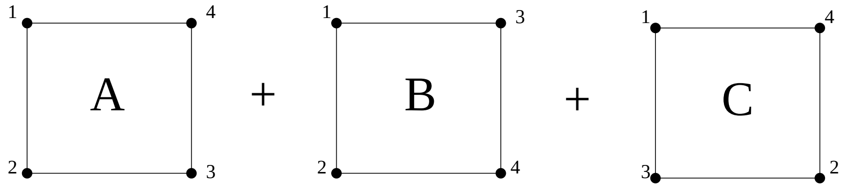

Figure 1: Connected part of the free four-point function of the spin-zero current J.

Sub-leading terms in the expansion of the box diagram should correspond to the contribution of the currents $J_l \sim \overrightarrow{\varphi} \cdot \partial^l \overrightarrow{\varphi}, l > 0$. This structure is precisely what we want to study in detail and the best way to identify all these contributions is via a conformal partial wave expansion (CPWE); i.e., decomposing into eigenfunctions of the quadratic Casimir of the conformal group $\mathrm{SO}(1, d+1)$ in Euclidean space $\mathbb{R}^d$ [12].

4. From free fields to AdS via CPWE

The attempts to cast the box diagrams into AdS amplitudes via Schwinger parametrization have not succeeded so far [4]. From the previous analysis of the OPE, it becomes apparent that the whole tower of HS field exchange has to be taken into account. Even though some progress has been made in obtaining bulk to bulk propagators for the HS fields in AdS [13], there is no closed analytic form that could be used to include all the infinite tower. We will content ourselves with the CPW amplitude to mimic the effect of the corresponding exchange Witten graph. In general, the CPW is contained in the exchange Witten graph but there appear additional terms that cannot be precisely identified as CPWs [14–18].

Let us first quote the essentials of the CPWE (see, e.g., [18] and references therein). The contribution of a quasi-primary $O^{(l)}_{\mu_1 \ldots \mu_l}$ of scale dimension Δ and spin l, and its derivative descendants, to the OPE of two scalar operators ϕ_i of dimension Δ_i,

$$\phi_1(x)\phi_2(y) \sim \frac{C_{\phi_1\phi_2 O^{(l)}}}{C_{O^{(l)}}} \frac{1}{\mid x-y \mid^{\Delta_1+\Delta_2-\Delta+l}} C^{(l)}(x-y, \partial_y)_{\mu_1\ldots\mu_l} O^{(l)}_{\mu_1\ldots\mu_l}(y). \tag{4.1}$$

The derivative operator is fixed by requiring consistency of the OPE with the two- and three-point functions of the involved fields (see appendix A). Based on these constraints one can work out the contribution of the conformal block corresponding to the quasi-primary $O^{(l)}$, and its derivative descendants, to the four-point function. This is given by the Conformal Partial Wave (see appendix C)

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle \sim \frac{C_{\phi_1\phi_2 O^{(l)}} C_{\phi_3\phi_4 O^{(l)}}}{C_{O^{(l)}}} \left(\frac{r_{24}}{r_{14}}\right)^{\Delta_{12}/2} \left(\frac{r_{14}}{r_{13}}\right)^{\Delta_{34}/2} \frac{u^{(\Delta-l)/2}}{(r_{12}r_{34})^{\Delta_\phi}} \\ \times G^{(l)}\left(\frac{\Delta-\Delta_{12}-l}{2}, \frac{\Delta+\Delta_{34}-l}{2}, \Delta; u, v\right),\end{aligned} \tag{4.2}$$

which depends on the two cross-ratios

$$u = \frac{r_{12}\, r_{34}}{r_{13}\, r_{24}}, \qquad v = \frac{r_{14}\, r_{23}}{r_{13}\, r_{24}}. \tag{4.3}$$

The CPWs have been obtained as double series in the direct channel limit $(u, 1-v) \to 0$ by several authors (see, e.g. [19] and references therein). They can also be shown to satisfy the recurrence relation (C.7), obtained by Dolan and Osborn [18].

4.1 CPWs of the HS free currents

We are interested in the singlet bilinears, minimal twist operators, in the free O(N) vector model. These are the spin-zero ($J \sim \varphi^a \varphi^a$) and the higher spin conserved currents ($J_l \sim \varphi^a \partial^l \varphi^a$) with canonical dimension $\Delta_l = d - 2 + l$. We can consider this limiting case in the recurrences C.7 by first setting $e = (\Delta - l)/2 = d/2 - 1 = \delta$ and in the end $S = 2\delta + l$, $b = \delta$. A crucial simplification occurs in the recurrence relation, only the third line in (C.7) survives:

$$G^{(l)}(b, \delta, S; u, v) = \frac{1}{2} \frac{S + l - 1}{\delta + l - 1} \left\{ G^{(l-1)}(b, \delta, S; u, v) - G^{(l-1)}(b + 1, \delta, S; u, v) \right\} \tag{4.4}$$

The iteration can then be easily done for the coefficients of the double expansion

$$G^{(l)}(b, e, S; u, v) = \sum_{m,n=0}^{\infty} a_{\rm nm}^{(l)}(b, S) \, \frac{u^n}{n!} \frac{(1 - v)^m}{m!} \, . \tag{4.5}$$

Pascal's triangle coefficients $\binom{l}{k}$ arise to get

$$a_{\rm nm}^{(l)}(b, S) = \frac{1}{2^l} \frac{(S)_l}{(\delta)_l} \sum_{k=0}^{l} (-1)^k \binom{l}{k} a_{\rm nm}^{(0)}(b + k, S), \tag{4.6}$$

where $\delta = \mu - 1 = d/2 - 1$ and we start with the scalar exchange C.8

$$a_{\rm nm}^{(0)}(b, S) = \frac{(\delta)_{m+n}}{(S)_{m+2n}} (S - b)_n (b)_{m+n}. \tag{4.7}$$

This can be summed up into a closed form involving a terminating generalized hypergeometric of unit argument

$$a_{\rm nm}^{(l)}(b, S) = \frac{1}{2^l} \frac{(\delta + l)_{m+n-l}}{(S + l)_{m+2n-l}} (b)_{m+n} (S - b)_n \; {}_3F_2\left(\begin{matrix} -l, 1 + b - S, b + m + n \\ b, 1 + b - S - n \end{matrix} \right). \tag{4.8}$$

With these conventions, the normalization is fixed by

$$a_{0l}^{(l)} = (-\frac{1}{2})^l l!. \tag{4.9}$$

Now, using twice an identity ([20], pp.141), obtained as a limiting case of a result due to Whipple for balanced ${}_4F_3$ series, one can rewrite the coefficients as a terminating (after $n + 1$ terms) series. This coincides with the result from the "Master Formula" in [21][6] for $b = \delta$ and $S = 2\delta + l$

$$a_{\rm nm}^{(l)} = a_{0l}^{(l)} \binom{m + n}{l} \frac{(\delta + l)^2_{m+n-l}}{(2\delta + 2l)_{m+n-l}} \; {}_3F_2\left(\begin{matrix} -n, 1 + m + n, \delta + m + n \\ 1 + m + n - l, 2\delta + m + n + l \end{matrix} \right)$$

[6] In their normalization, $a_{0l}^{(l)}$ are set to 1.

$$= a_{0l}^{(l)} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{m+n+s}{l} \frac{(\delta+l)_{m+n-l}(\delta+l)_{m+n-l+s}}{(2\delta+2l)_{m+n-l+s}}. \tag{4.10}$$

In this form one can easily recognize a triangular structure of the coefficients, i.e. $a_{\rm nm}^{(l>m+2n)} = 0$, which has proved useful in performing computer symbolic algebraic manipulations [21].

4.2 Modified CPWE: closure of the singlet bilinear sector

Now we compute the contribution of the singlet bilinear sector to the four-point function by summing the CPWs with the corresponding fusion coefficients ($\gamma_l^{\rm uv}$) in terms of those of the two and three-point functions[7] as (see 4.2)

$$(\gamma_l^{\rm uv})^2 \; = \; C^2_{JJJ_l}/C_{J_l}. \tag{4.11}$$

Using our previous results (3.9) and (3.12) we have

$$(\gamma_l^{\rm uv})^2 \; = \; 16N \, \frac{2^l}{l!} \, \frac{2\,(\delta)_l^2}{(2\delta-1+l)_l}. \tag{4.12}$$

The result of the direct channel summation (appendix D) is the observation that we can expand the first two terms (boxes) of (3.13) in partial waves of the bilinears, in the s-channel, as (D.9)

$$16N \, \frac{u^\delta}{(r_{12}r_{34})^{2\delta}} \left\{ 1 + v^{-\delta} \right\} = \frac{1}{(r_{12}r_{34})^{2\delta}} \sum_{l\geq 0,\, even} (\gamma_l^{\rm uv})^2 u^{(\Delta_l - l)/2} G^{(l)}(\delta,\delta,\Delta_l;u,v). \tag{4.13}$$

The full connected correlator is obtained then by crossing symmetry, since the three box diagrams A,B,C transform under crossing symmetry in the following way,

$$(2\to 4, t-channel)(u,v) \to (v,u) : (A,B,C) \to (A,C,B) \tag{4.14}$$

$$(2\to 3, u-channel)(u,v) \to (1/u, v/u) : (A,B,C) \to (C,B,A). \tag{4.15}$$

What we have found amounts to the diagrammatic identity in figure 2.

Our rewriting is different from the standard CPWE where the whole crossing symmetric result is reproduced in each channel. The OPE of two scalar bilinear J contains the contributions of the identity, of the conformal blocks of the bilinears (minimal twist) and also of the "double-trace" (higher twist) operators starting with $(\overrightarrow{\varphi}^2)^2$. In the large N analysis, one can see that when the OPE is inserted in the four-point function, the identity produces only one piece of the disconnected part (which goes as N^2) and the completion comes precisely form the double-traces (their fusion coefficients squared also goes as

[7] We can check the consistency of our conventions by comparing for the energy-momentum tensor ($l=2$). To keep track of the normalization coming from Ward identities we use $\phi = \frac{1}{\sqrt{2N}} J$ and the canonically normalized energy-momentum tensor $T = -\frac{1}{2(d-1)S_d} J_2$ [18], where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$, to have $C_{\phi\phi T} = -\frac{\Delta_\phi d}{d-1} = -\frac{(d-2)d}{d-1}$. The fusion coefficient in eq. (4.13) gets multiplied by $(\frac{1}{\sqrt{2N}})^4$. Then one gets from eq. (4.12) for the coefficient of the energy-momentum two-point function $\frac{C_T}{S_d^2}$, with the well known result for the free O(N) vector model $C_T = \frac{d}{d-1}N$.

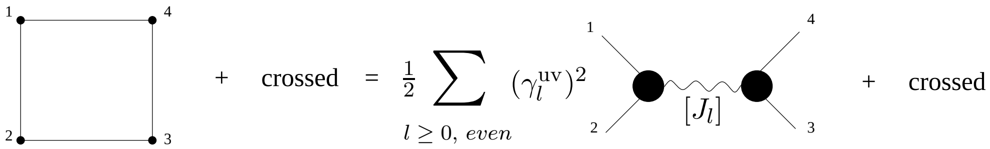

Figure 2: Free four-point function as a sum partial waves of minimal twist operators.

$N^2 + O(N)$) [3, 22]. It is also easy to see that the bilinear sector only contributes to the connected part (which goes as N, just like the fusion coefficients squared of this minimal twist sector, eq. 4.12).

Now the full disconnected piece is obtained from the Witten graphs containing two disconnected lines, where the three channels are included and with no need of additional fields in the bulk of AdS. One would then expect that for the connected part, something similar might happen. At leading $1/N$, tree approximation for the bulk theory, we have a classical field theory where one has to consider the exchange graphs in each channel separately; trading bulk exchanges by the corresponding CPW, one should then expect to write the connected part in terms of only CPWs of the minimal twist sector in the three channels, with no explicit reference to contributions from higher-twist/double-trace operators. To our surprise, this is precisely what we have obtained above!

So that in this way, we rescue the closure of the minimal twist sector that is in correspondence with a consistent truncation to the massless sector of the dual HS bulk theory. This result is valid as well for the bilinear single trace sector of free gauge theories considered in [1, 2, 4, 23], and amounts to a closure of the twist-two sector (without the double-trace operators this time!) in $d = 4$ by including the crossed channels, in conformity with the expectations for a consistent truncation of the bulk theory [2, 23].

5. Degeneracy of the hologram: IR CFT at d=3

Now we examine a peculiarity of this O(N) vector model at $d = 3$, which mimics the effect of double-trace deformations of the free gauge theory (see e.g. [6] and references therein).

The canonical dimension of the scalar J is $\Delta = 1$. This value Δ_- is mapped, via AdS/CFT Correspondence, to a conformally coupled bulk scalar. However,there is a conjugate dimension $\Delta_+ = 2$ which agrees (at leading $1/N$ order)with the known result for the dimension of J at the interacting IR critical point. This led Klebanov and Polyakov to conjecture that the minimal bosonic HS gauge theory with even spins and symmetry group $hs(4)$ is related, via standard AdS/CFT methods with the "conventional" branch Δ_+, to the interacting large N vector model at its IR critical point. The free theory, UV fixed point, corresponds then to the other branch Δ_-.

The existence of this IR stable critical point of the O(N) vector model below four dimension is a well established fact.[8] Standard approaches are the ϵ-expansion in $4-\epsilon$ dimensions which leads to the Wilson-Fisher fixed point and the large-N expansion which reveals a fixed point at $2 < d < 4$. Our analysis will be restricted to the leading $1/N$ results. An efficient way to perform the large-N expansion is introducing an auxiliary field α coupled to the vector field via a triple vertex $\alpha\,\varphi^a\varphi^a$ and then integrate out φ^a which appears now quadratically, to get the effective action for α. The diagrammatic expansion in $1/N$ involves skeleton graphs with the field φ^a running along internal lines and the triple vertices of two φ's with the auxiliary field [25].

5.1 IR two- and three-point functions

At leading $1/N$ we keep the free two-point function of the elementary fields φ^a, they acquire anomalous dimension of order $1/N$, and for the auxiliary field α with dimension $\Delta_+ = 2$ one can set[9]

$$\langle\alpha(x)\alpha(0)\rangle = r^{-2}, \tag{5.1}$$

absorbing the normalizations in the vertex, which becomes [26](see appendix E for notations)

$$(\frac{z_1}{N})^{1/2} \quad , \quad z_1 = -2p(2). \tag{5.2}$$

Analogously, the three-point function form is dictated by the conformal symmetry

$$\langle\alpha(x_1)\alpha(x_2)\alpha(x_3)\rangle = C_{\alpha\alpha\alpha}\,(r_{12}r_{23}r_{31})^{-1} \tag{5.3}$$

and

$$\langle\alpha(x_1)\alpha(x_2)J_{l\;\nu_1\ldots\nu_l}(x_3)\rangle = C_{\alpha\alpha J_l}\;r_{12}^{-2+\delta}(r_{23}r_{31})^{-\delta}\;\lambda^{x_3}_{\mu_1\ldots\mu_l}(x_1x_2). \tag{5.4}$$

They are related to the respective free correlators by amputation relations (Legendre transform). For the two-point function

$$\langle J(x)J(0)\rangle = -\frac{2N}{p(2)}\langle\alpha(x)\alpha(0)\rangle^{-1}. \tag{5.5}$$

With our choice of normalizations, we have then to amputate with $\langle\alpha(x)\alpha(0)\rangle^{-1}$ and multiply by the factor $(-\frac{2N}{p(2)})^{\frac{1}{2}}$ for each scalar leg that is amputated to go from IR to UV at leading N, while the legs corresponding to the HS current remain the same. To compare this normalization with that of Klebanov and Witten [9] and appendix F, we denote their Legendre transformation by $\Gamma[A] = W[\phi_0] - (2\Delta_+ - d)\int A\phi_0$. Then at $d=3$ the source for α is ϕ_0/π and that for J is $A/(2\pi\sqrt{N})$. The amputation is done with the D'EPP formula and its generalization (see appendix E). For the three-point function of the scalars one gets

$$C_{\alpha\alpha\alpha} = N(-\frac{2}{Np(2)})^{\frac{3}{2}}v^2(2,\delta,\delta)v(2,1,2\delta-1) \tag{5.6}$$

[8] In four dimensions the IR fixed point merges with the UV one and the duality is no longer valid in the way we have just presented. Still one can modify the O(N) Vector Model (by gauging) to have a similar holographic scenario [24].

[9] In what follows and in an abuse of notation, a correlator involving α is understood to be computed at the IR fixed point, while the same correlator at UV contains J instead.

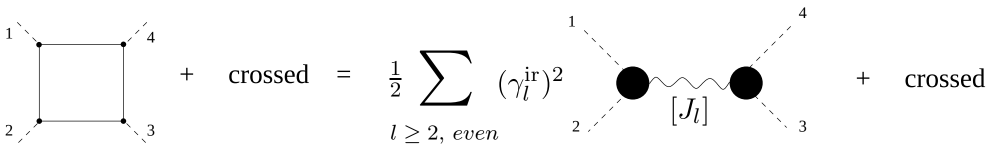

Figure 3: IR four-point function as a sum of partial waves of minimal twist operators at $d = 3$.

as obtained in [26, 28]. There is a factor $\frac{1}{\Gamma(d-3)}$ that forces the vanishing of the IR three-point function at $d = 3$ in correspondence with the vanishing of the bulk coupling in the HS AdS_4 theory [28, 29].

We extend this amputation procedure to the other three-point functions to get[10]

$$C_{\alpha\alpha J_l} = 2^{l+1} \frac{l!\,(2\delta - 1)\,(\delta)_l}{(2\delta - 1)_l} \tag{5.7}$$

which agrees with what was obtained in [10][11] by a different procedure, namely computing the four-point function first of the two scalars with two elementary fields and then forming the HS current by contracting the two legs of the elementary fields acting with derivatives and letting their argument to coincide at the end. This computation was done at $d = 3$, however we corroborate the validity for any $2 < d < 4$. This has the surprising implication that a graph contributing to the four-point function above mentioned, which vanishes at $d = 3$, does not contribute to the HS current correlator at generic $2 < d < 4$ as well.

5.2 Scalar four-point function and fusion coefficients at IR

Let us now examine the implications of the degeneracy of the hologram for the four-point function at the IR critical point. The AdS amplitude should involve the same bulk exchange graphs, only the scalar bulk-to-boundary and and bulk-to-bulk propagators are switched to the ones with Δ_+. We trade them by CPWs with the appropriate fusion coefficients that follow from the amputation program. Therefore we guess the modified CPWE in the interacting theory as indicated in figure 3,

The fusion coefficients

$$(\gamma_l^{\rm ir})^2 \;=\; C^2_{\alpha\alpha J_l}/C_{J_l}, \tag{5.8}$$

using our previous results from the amputations, are given by

$$(\gamma_l^{\rm ir})^2 \;=\; \frac{1}{N}\,\frac{2^l}{l!}\,\frac{8(l!)^2\,(\delta)_l^2}{(2\delta)_{l-1}\,(2\delta)_{2l-1}}. \tag{5.9}$$

That the four-point function at the IR critical point at leading $1/N$ has precisely this expansion has been shown by Rühl [30], by explicit computations at the IR critical point and the fusion coefficients obtained by extrapolation of computer algebraic manipulations.

[10]This time the amputation is done with the generalization E.5 of the D'EPP formula.

[11]There is a relative factor of 2 due to normalization of the HS current and a missing factor 2^l, by misprint, in equation (97) of this paper.

What we have analytically found confirms those results and prove their validity for the whole range $2 < d < 4$ where the scalar contribution accounts for the one-line-reducible graph, both of them being now non-vanishing. The shadow contribution in the one-line-reducible graph is canceled by contributions from the box as shown in [3] for the leading singular term and in [21] for the full CPW. The quotient $\gamma_l^{\mathrm{ir}}/\gamma_l^{\mathrm{uv}}$ for $d = 3$ is

$$\gamma_l^{\mathrm{ir}}/\gamma_l^{\mathrm{uv}} = l/(2N)\,, \tag{5.10}$$

which is valid even for $l = 0$ since $\gamma_l^{\mathrm{ir}} = 0$. Note that in the other normalization for fields with sources ϕ_0 and A, the ratio turns out to be equal to $2l$.

6. CPWs vs. AdS exchange graphs at $d = 3$

The two and three-point functions considered before can be reproduced from a bulk action, being relevant only up to cubic terms of the bulk Lagrangian. These have been obtained in [10], assuming a bulk coupling of the HS field with a bulk current, bilinear in the scalar bulk field and involving up to l derivatives.[12] In their scheme there are two different couplings of the HS bulk field to two bulk scalars, one to reproduce the UV correlators and another for the IR case, and therefore two different bulk Lagrangians. We however adopt the view of a unique bulk Lagrangian, as expected from double trace deformations. It is not difficult then to realize that the bulk graphs corresponding to the coupling of the HS fields to the AdS current, bilinear in the bulk scalar, obtained in [10] lead to boundary three-point functions which are precisely related by amputation of the scalar legs. This is done in the boundary theory with the generalized D'EPP formula and in the bulk graph this amounts to changing the dimension $\Delta_- \leftrightarrow \Delta_+$ of the bulk-to-boundary propagator of the scalar legs [27].

For the four-point function, the CPW expansion obtained is indeed a step in the ambitious program of bottom to top approach, in which one uses the knowledge of the boundary CFT to reconstruct the bulk theory. This is a formidable task, but a Witten graph is certainly closer to a CPW as we know since the early days of AdS/CFT, although the precise correspondence has always been elusive and tricky [14, 15].

Here we will study the scalar exchange and see what happens when one considers the boundary scalar bilinear to have canonical dimension $\Delta_- = d - 2 = 1$.

Let us start with the free three-point function. Despite the success of predicting the vanishing of the scalar three-point function at the IR critical point and matching with the HS bulk theory [28, 29], the non-vanishing result for the free correlator cannot be obtained from a null result via the proposed Legendre transformation. One is forced to make a regularization, and the appropriate way turns out to be that the bulk coupling goes like $g \sim (d - 3)$. Here we propose to compute directly with the canonical dimension and to control the divergence of the Witten graph by dimensional regularization. The graph is

[12]However, what one obtains from the HS theory for $l = 2$ is a bulk energy-momentum involving infinitely many derivatives. It is still an open issue to see whether both formulations are equivalent via some field redefinition. This we believe must first be clarified before trying to explore HS bulk exchange graphs, the coupling to the scalar is still ambiguous although their should be fixed by the conformal symmetry.

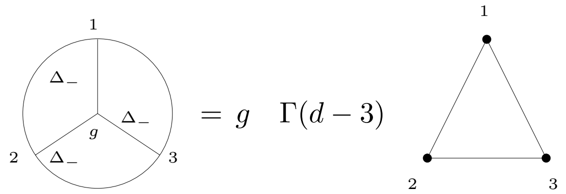

Figure 4: Free three-point function and star Witten graph at $d \to 3$.

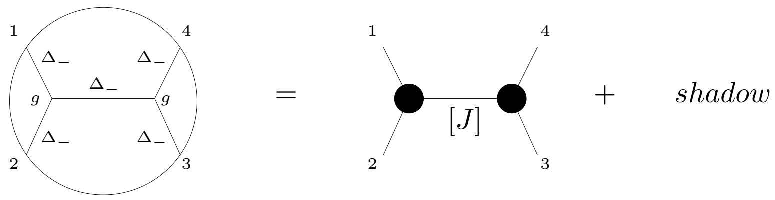

Figure 5: Scalar exchange Witten graph VS. scalar CPWs in the limit $d \to 3$

divergent at $d = 3$ (see E.6, E.7), but a cancellation against the vanishing coupling gives the correct result for the free correlator.[13]. Starting with the free correlator and following Gopakumar [4] in bringing it to an AdS Witten graph, one gets the identity sketched[14] in figure 4 . The divergence of the star graph is then controlled by the zero in $1/\Gamma(d-3)$, rendering the correct result for the free correlator.

Now we move on and examine the scalar exchange Witten graph, with the external legs having canonical dimension $\Delta_- = d-2 \to 1$ and coupling vanishing as $g \sim (d-3) \to 0$. A suitable evaluation of this graph, worked out in [17] using Mellin-Barnes representation and performing a contour integral, is given by a double series expansion involving three coefficients $a^{(1)}_{\rm nm}$, $b^{(1)}_{\rm nm}$ and $c^{(1)}_{\rm nm}$ (in [17], eq.23-26). In the limit $\Delta = \widetilde{\Delta} = d-2$ and $d \to 3$, they all become divergent but only $c^{(1)}_{\rm nm}$ and the first term in $b^{(1)}_{\rm nm}$ develop a double pole, the rest being less singular. They cancel against the double zero from g^2 and the final result can be precisely casted into the CPW of the free scalar J plus its shadow, a scalar of dimension $\Delta_+ = 2$. The piece coming from the $c^{(1)}_{\rm nm}$ coefficient goes to the $c_{\rm nm}(1)$ coefficient of the CPW and the contribution from $b^{(1)}_{\rm nm}$ produces the shadow term with coefficient $c_{\rm nm}(2)$ ([17], eq.35-36). We end up with a precise identification in term of CPWs as sketched in figure 5.

When one continues the crossed channel expansions to get their contribution in the direct channel one gets $log\, u$ terms but no non-analytic terms in $(1-v)$.[15] This happens

[13]Essentially the same cancellation that occurs for extremal correlators in standard AdS/CFT.

[14]In this section the equality sign is to be understood modulo finite factors that we omit for simplicity. The precise relation can be read from E.6, E.7.

[15]This is a simple way to see that the bold identification of the scalar exchange Witten graph with the CPW, as originally proposed in [14], was certainly not correct. In our case we bypass this difficulty due to

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both for a Witten graph and for the combined CPW , i.e. direct plus shadow. The mechanisms that prevent the appearance of such terms are different [16, 17], in one case is due to some nontrivial hypergeometric identities and in the second case is due to the presence of the shadow field contribution. In the above case, both mechanisms coincide and the identification in terms of CPW is precise (this identification is in general incomplete, as mentioned before). That is, there is more structure in the scalar exchange graph than in a generic one and we take this as a good sign that after all, when computing the infinite tower of exchange diagrams, many delicate cancellations of additional terms take place to end up with just the sum of CPWs as obtained before. In particular, the additional term in the scalar case is a *shadow* contribution which are indeed absent in any full physical amplitude.

7. Conclusion

We have re-organized the CPWE of the free four-point function of the scalar singlet bilinear in the natural way one would expect from AdS/CFT Correspondence; that is, by explicit inclusion of the crossed channels and involving only CPWs of the minimal twist sector which is holographically dual to the bulk HS gauge fields. This result is applicable as well to the corresponding sector of free gauge theories. Kinematically, double-trace operators are dual to two-particle bulk states; however, it is hard to see how such bulk states arise in the tree bulk computation that one has to perform at leading large-N. We guess that the double-trace operators arise indirectly, just in the way they show up in the free O(N) vector model.

In $2 < d < 4$ dimensions, one can flow (at leading large-N) into the IR fixed point of the O(N) vector model by just Legendre transforming. In this way, we have completed the program initiated in [28] for the three-point functions. In addition, under the assumption of a degenerate hologram, i.e. same bulk content but different asymptotics for the scalar bulk field, the modified CPWE of the four-point function was also obtained at IR.

All two- and three-point function coefficients as well as fusion coefficients were analytically obtained, in some cases corroborating extrapolations from computer algebraic manipulations.

For the scalar exchange Witten graph with canonical dimensions, a funny cancellation occurs and the result can be precisely identified in terms of CPWs of the corresponding scalar and its shadow. This reveals more structure than the generic case, and we hope that such "accidents" are indeed needed to obtain the full four-point correlator if one were able to sum the infinite tower of HS field exchanges.

Acknowledgments

We thank F.A. Dolan, R. Gopakumar, H. Osborn and W. Rühl for valuable e-mail correspondence. We are also grateful to A. Torrielli for useful discussions. This work was partially supported by DFG via GRK 271 and DO 447/3-3.

the shadow term, which makes the whole expression manifestly "shadow-symmetric".

A. Restrictions from conformal invariance

Conformal invariance dictates the form the three-point function of two scalars, of dimension Δ_i, with a totally symmetric traceless rank l tensor, of dimension Δ, to be (see e.g. [31])

$$\begin{aligned}&\langle \phi_1(x_1)\phi_2(x_2)O^{(l)}_{\mu_1\ldots\mu_l}(x_3)\rangle \\ &= C_{\phi_1\phi_2 O^{(l)}} \frac{1}{r_{12}^{(\Delta_1+\Delta_2-\Delta+l)/2} r_{13}^{(\Delta+\Delta_{12}-l)/2} r_{23}^{(\Delta-\Delta_{12}-l)/2}} \lambda^{x_3}_{\mu_1\ldots\mu_l}(x_1,x_2),\end{aligned} \tag{A.1}$$

where

$$\lambda^{x_3}_{\mu_1\ldots\mu_l}(x_1,x_2) = \lambda^{x_3}_{\mu_1}(x_1,x_2)\ldots\lambda^{x_3}_{\mu_l}(x_1,x_2) - traces, \quad \lambda^{x_3}_{\mu}(x_1,x_2) = \left(\frac{x_{13}}{r_{13}} - \frac{x_{23}}{r_{23}}\right)_\mu, \tag{A.2}$$

and $\Delta_{\rm ij} = \Delta_i - \Delta_j$.

Also the form of the two-point function of the symmetric traceless rank l tensor, which defines an orthogonality relation with respect to spin and conformal dimension, is required to be (see e.g. [31])

$$\langle O^{(l)}_{\mu_1\ldots\mu_l}(x_1) O^{(l)}_{\nu_1\ldots\nu_l}(x_2)\rangle = C_{O^{(l)}} \frac{1}{r_{12}^{\Delta}}\; sym\{I_{\mu_1\nu_1}(x)\ldots I_{\mu_l\nu_l}(x)\} \tag{A.3}$$

where

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{r} \tag{A.4}$$

is the inversion tensor, related to the Jacobian of the inversion $x_\mu \to x_\mu/r$, and sym means symmetrization and removal of traces.

The structure of the general four-point conformal correlator is required to be

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \prod_{i<j,1}^{4} (r_{\rm ij})^{(\Sigma/3-\Delta_i-\Delta_j)/2} F(u,v), \tag{A.5}$$

where $\Sigma = \Delta_1 + \cdots + \Delta_4$ and F is an arbitrary function of the invariant ratios.

B. HS two-point function coefficient

The double sum can be cast into the form

$$\frac{1}{4}\sum_{k=0}^{l}(-1)^k\binom{l}{k}\frac{(\delta)_l\,(\delta)_{2l-k}}{(\delta)_{l-k}}\sum_{s=0}^{l}(-1)^s\binom{l}{s}\frac{(\delta+k)_s\,(\delta+2l-k)_{-s}}{(\delta)_s\,(\delta+l)_{-s}}. \tag{B.1}$$

The last sum can be transformed, using elementary identities such as $(-1)^k\binom{n}{k} = \frac{(-n)_k}{k!}$ and $(-z)_n = (-1)^n\frac{1}{(1+z)_{-n}}$, in a terminating generalized hypergeometric series ${}_3F_2$ of unit argument

$$\sum_{s=0}^{l}\frac{1}{s!}(-l)_s\frac{(\delta+k)_s\,(1-\delta-l)_s}{(\delta)_s\,(1-\delta-2l+k)_s} = {}_3F_2\left(\begin{matrix}-l,\delta+k,1-l-\delta\\ \delta,1-2l-\delta-k\end{matrix}\right). \tag{B.2}$$

The evaluation of ${}_3F_2$ can be done by applying twice the same identity used in eq. (4.8),

$$ {}_3F_2\binom{-n,a,b}{d,e} = \frac{(e-a)_n}{(e)_n}\,{}_3F_2\binom{-n,a,d-b}{d,1+a-n-e} \tag{B.3} $$

to get

$$ \begin{aligned} &\frac{(k-l)_l}{(1-2l-\delta+k)_l}\frac{(1-k+l)_k}{(1-k)_k}\,{}_3F_2\binom{-k,-l,2\delta+l-1}{\delta,-l} \\ &= \frac{(k-l)_l}{(1-2l-\delta+k)_l}\frac{(1-k+l)_k}{(1-k)_k}\frac{(1-\delta-l)_k}{\delta_k} = (-1)^k\frac{l!}{(\delta)_k}(\delta+l)_{l-k} \end{aligned} \tag{B.4} $$

where the ${}_3F_2$ reduced to an ordinary ${}_2F_1$ of unit argument evaluated with the Chu-Vandermonde formula [20].

The sum that remains to be done reduces then again to a terminating ordinary hypergeometric of unit argument that is evaluated as before

$$ \begin{aligned} \frac{1}{4}\,l!\,(\delta)_l\sum_{k=0}^{l}\binom{l}{k}\frac{1}{(\delta)_k\,(\delta+l)_{-k}} &= \frac{1}{4}\,l!\,(\delta)_l\,{}_2F_1\binom{-l,1-\delta-l}{\delta} \\ &= \frac{1}{4}\,l!\,(2\delta-1+l)_l. \end{aligned} \tag{B.5} $$

C. CPW recurrences

Inserting the OPE[16] (4.1) and using the orthogonality relation (A.3) we have for the action of the derivative operator

$$ \begin{aligned} &\langle\phi_1(x_1)\phi_2(x_2)\,O^{(l)}_{\mu_1\dots\mu_l}(x_3)\rangle \\ &= \frac{C_{\phi_1\phi_2O^{(l)}}}{C_{O^{(l)}}}\frac{1}{r_{12}^{(\Delta_1+\Delta_2-\Delta+l)/2}}C^{(l)}(x_{12},\partial_{x_2})_{\nu_1\dots\nu_l}\langle O^{(l)}_{\nu_1\dots\nu_l}(x_2)\,O^{(l)}_{\mu_1\dots\mu_l}(x_3)\rangle. \end{aligned} \tag{C.1} $$

Inserting now the OPE (4.1) in the scalar four-point function one gets for the contribution of $O^{(l)}$ and its descendants

$$ \langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle \sim \frac{C_{\phi_1\phi_2O^{(l)}}}{C_{O^{(l)}}}\frac{1}{r_{12}^{(\Delta_1+\Delta_2-\Delta+l)/2}} \tag{C.2} $$

$$ \times C^{(l)}(x_{12},\partial_{x_2})_{\mu_1\dots\mu_l}\langle O^{(l)}_{\mu_1\dots\mu_l}(x_2)\,\phi_3(x_3)\phi_4(x_4)\rangle. \tag{C.3} $$

In order to be able to act as before with the derivative operator on a two-point function, one has to re-write the x_2-dependence in $\langle O^{(l)}_{\mu_1\dots\mu_l}(x_2)\;\phi_3(x_3)\phi_4(x_4)\rangle$ in a suitable way. This is achieved by introducing the "shadow" operator (conformal partner) $O^{*(l)}$, a "conventional" operator with labels $(\Delta^*,l)=(d-\Delta,l)$

[16] The OPE involves the sum over the complete set of quasi-primaries. We consider no degeneracies for simplicity, ie. no additional labels apart from (Δ,l).

$$O^{(l)}_{\mu_1\ldots\mu_l}(x_2) = \int d^dx \, \langle O^{(l)}_{\mu_1\ldots\mu_l}(x_2) \, O^{(l)}_{\nu_1\ldots\nu_l}(x)\rangle \, O^{*(l)}_{\nu_1\ldots\nu_l}(x). \tag{C.4}$$

Inserting this relation and using (C.1) one gets

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle \sim \int d^dx \, \langle\phi_1(x_1)\phi_2(x_2) \, O^{(l)}_{\mu_1\ldots\mu_l}(x)\rangle \, \langle O^{*(l)}_{\mu_1\ldots\mu_l}(x) \, \phi_3(x_3)\phi_4(x_4)\rangle. \tag{C.5}$$

In all, one has just inserted the projection operator [3, 14]

$$\mathcal{P}_l \;=\; \int d^dx \, O^{(l)}_{\mu_1\ldots\mu_l}(x)|0\rangle \, \langle 0|O^{*(l)}_{\mu_1\ldots\mu_l}(x). \tag{C.6}$$

The integrand can be cast into a form involving Gegenbauer polynomials after contraction of Lorentz indices, and using their recurrence relations one gets the following recurrences[17] [18]

$$\begin{aligned}
&G^{(l)}(b,e,S;u,v)\\
&= \frac{1}{2}\frac{S+l-1}{d-S+l-2}\left\{\frac{d/2-e-1}{f+l-1}\Big(vG^{(l-1)}(b+1,e+1,S;u,v) - G^{(l-1)}(b,e+1,S;u,v)\Big)\right.\\
&\qquad\left.+\frac{d/2-f-1}{e+l-1}\Big(G^{(l-1)}(b,e,S;u,v) - G^{(l-1)}(b+1,e,S;u,v)\Big)\right\}\\
&-\frac{1}{4}\frac{(S+l-1)(S+l-2)}{(d-S+l-2)(d-S+l-3)}\frac{(d/2-e-1)(d/2-f-1)}{(f+l-1)(e+l-1)}\frac{(l-1)(d+l-4)}{(d/2+l-2)(d/2+l-3)}\\
&\qquad uG^{(l-2)}(b+1,e+1,S;u,v),
\end{aligned} \tag{C.7}$$

with $S = e+f+l$. The starting point is the scalar result that in the direct channel limit, $u, 1-v \sim 0$ is given by the double power expansion

$$G^{(0)}(b,e,S;u,v) = \sum_{m,n=0}^{\infty} \frac{(S-b)_n(S-e)_n}{(S+1-d/2)_n}\frac{(b)_{n+m}(e)_{n+m}}{(S)_{2n+m}}\frac{u^n}{n!}\frac{(1-v)^m}{m!}. \tag{C.8}$$

D. UV fusion coefficients

Using the double expansion in the direct channel limit $u, 1-v \sim 0$, the sum we have to perform on the r.h.s. of (4.13) is

$$\sum_{l\geq 0,\, even} (\gamma^{\rm uv}_l)^2 \, a^{(l)}_{\rm nm} \tag{D.1}$$

with $a^{(l)}_{\rm nm}$ and $(\gamma^{\rm uv}_l)^2$ given in (4.10) and (4.12), respectively.

First one can perform the sum over l, due to the triangular structure the sum is up to $m+2n$, and sum over all $l's$ writing

$$(\gamma^{\rm uv}_l)^2 = [1+(-1)^l]\frac{{(\delta)_l}^2}{(2\delta+l-1)_l}\frac{16N}{a^{(l)}_{0l}}. \tag{D.2}$$

[17] In fact, by the previous procedure one gets in addition the contribution from the shadow operator. What follows is valid for the direct contribution.

Now, after straightforward manipulations,[18] the sum can be casted in terms of a terminating well-poised generalized hypergeometric ${}_3F_2$ of argument ± 1 as

$$\sum_{l\geq 0,\, even} (\gamma_l^{\rm uv})^2\, a_{\rm nm}^{(l)} = 16N\,(\delta)_{m+n} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{(\delta)_{m+n+s}}{(2\delta)_{m+n+s}}$$

$$\times \left\{ {}_3F_2^{-}\begin{pmatrix} -m-n-s, \delta+\frac{1}{2}, 2\delta-1 \\ 2\delta+m+n+s, \delta-\frac{1}{2} \end{pmatrix} + {}_3F_2^{+}\begin{pmatrix} -m-n-s, \delta+\frac{1}{2}, 2\delta-1 \\ 2\delta+m+n+s, \delta-\frac{1}{2} \end{pmatrix} \right\}. \tag{D.3}$$

The evaluation at -1, with the particular case of a corollary of Dougall's formula ([20], pp.148)

$${}_3F_2^{-}\begin{pmatrix} a, 1+\frac{a}{2}, b \\ \frac{a}{2}, 1+a-b \end{pmatrix} = \frac{(1+a)_{-b}}{(\frac{1}{2}+\frac{a}{2})_{-b}} \tag{D.4}$$

gives

$$\frac{(2\delta)_{m+n+s}}{(\delta)_{m+n+s}}, \tag{D.5}$$

so that the first part is

$$16N\,(\delta)_{m+n} \sum_{s=0}^{n} (-1)^s \binom{n}{s} = 16N\,(\delta)_{m+n}\,\delta_{n,0} = 16N\,(\delta)_m\,\delta_{n,0}\,. \tag{D.6}$$

The evaluation at $+1$ done with Dixon's identity ([20], pp.72), which can also be derived from Dougall's formula,

$${}_3F_2^{+}\begin{pmatrix} a, b, c \\ 1+a-b, 1+a-c \end{pmatrix} = \frac{(1+a)_{-b}\,(1+a)_{-c}\,(1+\frac{a}{2})_{-b-c}}{(1+\frac{a}{2})_{-b}\,(1+\frac{a}{2})_{-c}\,(1+a)_{-b-c}} \tag{D.7}$$

produces a factor $(0)_{m+n+s}$ which vanishes for $m+n+s \neq 0$. At $m=n=0$ (this forces $s=0$) one gets 1, so that the second part contributes

$$16N\,(\delta)_{m+n}\,\delta_{n,0}\,\delta_{m,0} \;=\; 16N\,\delta_{n,0}\,\delta_{m,0}. \tag{D.8}$$

Finally, we find the equality

$$16N\,\delta_{n,0}\,\{\delta_{m,0} + (\delta)_m\} \;=\; \sum_{l\geq 0,\, even} (\gamma_l^{\rm uv})^2\, a_{\rm nm}^{(l)}, \tag{D.9}$$

which corresponds to the double expansion of equation 4.13.

E. D'EPP formula and star Witten graph

The inverse kernels are defined according to

$$p(\lambda) \int d^d x_3\, r_{13}^{-\lambda}\, r_{23}^{-d+\lambda} \;=\; \delta^d(x_{12}), \tag{E.1}$$

[18] A term $(2\delta+2l)_{-1}$ involving $2l$ must be casted into $\frac{1}{2}(\delta+l+\frac{1}{2})_{-1}$.

where

$$p(\lambda) = p(d-\lambda) = \pi^{-d}\frac{\Gamma(\lambda)\,\Gamma(d-\lambda)}{\Gamma(\frac{d}{2}-\lambda)\,\Gamma(\lambda-\frac{d}{2})}. \tag{E.2}$$

The D'Eramo-Parisi-Peliti formula [28, 31, 32] reads

$$\int d^d x_4\; r_{14}^{-\delta_1}\; r_{24}^{-\delta_2}\; r_{34}^{-\delta_3} \;=\; v(\delta_1,\delta_2,\delta_3)\; r_{12}^{-\frac{d}{2}+\delta_3}\;\; r_{23}^{-\frac{d}{2}+\delta_1}\;\; r_{13}^{-\frac{d}{2}+\delta_2}\,, \tag{E.3}$$

where $\delta_1+\delta_2+\delta_3 = d$ and

$$v(\delta_1,\delta_2,\delta_3) \;=\; \pi^{\frac{d}{2}}\frac{\Gamma(\frac{d}{2}-\delta_1)\,\Gamma(\frac{d}{2}-\delta_2)\,\Gamma(\frac{d}{2}-\delta_3)}{\Gamma(\delta_1)\,\Gamma(\delta_2)\,\Gamma(\delta_3)}. \tag{E.4}$$

We also need a generalization of D'EPP [31], obtained by differentiation,

$$\int d^d x_4\; r_{14}^{-\delta_1}\; r_{24}^{-\delta_2}\; r_{34}^{-\delta_3}\; \lambda^{x_1}_{\mu_1\ldots\mu_s}(x_4,x_2) =$$

$$\frac{(\frac{d}{2}-\delta_2)_s}{(\delta_1)_s}\; v(\delta_1,\delta_2,\delta_3)\; r_{12}^{-\frac{d}{2}+\delta_3}\;\; r_{23}^{-\frac{d}{2}+\delta_1}\;\; r_{13}^{-\frac{d}{2}+\delta_2}\;\; \lambda^{x_1}_{\mu_1.\mu_s}(x_3,x_2)\,. \tag{E.5}$$

The star Witten graph with scalar legs of generic dimensions Δ_i $(i=1,2,3)$ is given by (see e.g. [11])

$$\frac{a(\Delta_1,\Delta_2,\Delta_3)}{r_{12}^{(\Delta_1+\Delta_2-\Delta_3)/2}\, r_{13}^{(\Delta_1+\Delta_3-\Delta_2)/2}\, r_{23}^{(\Delta_2+\Delta_3-\Delta_1)/2}}, \tag{E.6}$$

where

$$a(\Delta_1,\Delta_2,\Delta_3) = \frac{1}{2\pi^d}\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2})\frac{\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1+\Delta_3-\Delta_2}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1}{2})}{\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})}. \tag{E.7}$$

F. Regularized kernels

The aim of this part is to fix notation and thereby to summarize the facts concerning the reconstruction of the bulk fields out of its two types of asymptotics along the line of [9, 33]. Our presentation contains some new elements, insofar as we exclusively rely on convergent position space integrals. From them we will be able to *derive* the analytic continuation rules which usually appear a posteriori to give meaning to naively divergent integrals.

In AdS_{d+1} the scalar on shell bulk field $\phi(x)$, with $x=(z,\vec{x}),\ z\geq 0$ denoting Poincaré coordinates, has the near boundary asymptotics,[19] see e.g. [9, 34]

$$\phi(x) \;=\; z^{\Delta_-}(\phi_0(\vec{x})+\mathcal{O}(z^2)) \;+\; z^{\Delta_+}(A(\vec{x})+\mathcal{O}(z^2))\,, \tag{F.1}$$

[19] In the following we assume a suitable rapid falloff of A and ϕ_0 for $|\vec{x}|\to\infty$.

where $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$. The standard bulk to bulk propagators obey ($\Delta = \Delta_\pm$)

$$\begin{aligned}(\Box_x - m^2) G_\Delta(x,x') &= -g^{-\frac{1}{2}}\ \delta(x,x')\,,\\ G_\Delta(x,x') &= z'^\Delta G^0_\Delta(x,\vec{x'}) + \mathcal{O}(z'^{\Delta+2})\ .\end{aligned} \tag{F.2}$$

Using (F.1), (F.2) and Gauss theorem one gets with fixed $z' > 0$ [35–37]

$$\begin{aligned}\phi(x) = \ &\textstyle\int d^d\vec{x'}\ \{(\Delta - \Delta_-)\phi_0(\vec{x'})\ G^0_\Delta(x,\vec{x'})\ z'^{\Delta_- + \Delta - d} + \mathcal{O}(z'^{\Delta_- + \Delta - d + 2})\\ &+ (\Delta - \Delta_+) A(\vec{x'})\ G^0_\Delta(x,\vec{x'})\ z'^{\Delta_+ + \Delta - d} + \mathcal{O}(z'^{\Delta_+ + \Delta - d + 2})\}\ .\end{aligned} \tag{F.3}$$

Since always $\Delta_+ \geq \frac{d}{2}$, for the choice $\Delta = \Delta_+$ both $\mathcal{O}$-terms go to zero for $z' \to 0$. Choosing instead $\Delta = \Delta_-$, the vanishing of both $\mathcal{O}$-terms requires $\Delta_- > \frac{d-2}{2}$, i.e. just the unitarity bound. Altogether for $\frac{d-2}{2} < \Delta_- < \frac{d}{2} < \Delta_+$ one gets

$$\phi(x) = \int d^d\vec{x'}\ \phi_0(\vec{x'})\ K_{\Delta_+}(x,\vec{x'}) = \int d^d\vec{x'}\ A(\vec{x'})\ K_{\Delta_-}(x,\vec{x'})\,, \tag{F.4}$$

with

$$K_{\Delta_\pm}(x,\vec{x'}) = (2\Delta_\pm - d) \lim_{z'\to 0} z'^{-\Delta_\pm}\ G_{\Delta_\pm}(x,x') = \frac{\Gamma(\Delta_\pm)}{\pi^{\frac{d}{2}}\Gamma(\Delta_\pm - \frac{d}{2})}\ \frac{z^{\Delta_\pm}}{(z^2 + (\vec{x} - \vec{x'})^2)^{\Delta_\pm}}\ . \tag{F.5}$$

The reconstruction of the asymptotics (F.1) from the first eq. in (F.4) is given by

$$\begin{aligned}&\int d^d\vec{x'}\ \phi_0(\vec{x'})\ K_{\Delta_+}(x,\vec{x'}) = z^{\Delta_-}\ \phi_0(\vec{x})\ (1 + \mathcal{O}(z^2) + \cdots + \mathcal{O}(z^{2k})) \\ &+ \frac{\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\Delta_+ - \frac{d}{2})}\ z^{\Delta_+}\ \left(\int d^d\vec{x'}\ \frac{\phi_0(\vec{x'}) - \phi_0(\vec{x}) - \cdots - \frac{((\vec{x'}-\vec{x})\partial)^{2k}}{(2k)!}\phi_0(\vec{x})}{|\vec{x'} - \vec{x}|^{2\Delta_+}} + \mathcal{O}(z^2)\right),\end{aligned} \tag{F.6}$$

where k is the largest integer smaller than $\Delta_+ - \frac{d}{2}$. Similarly one finds from the second representation of $\phi(x)$ in (F.4) for $\frac{d-2}{2} < \Delta_- < \frac{d}{2}$

$$\begin{aligned}&\int d^d\vec{x'}\ A(\vec{x'})\ K_{\Delta_-}(x,\vec{x'}) = z^{\Delta_+}\ A(\vec{x})\ (1 + \mathcal{O}(z^{2(\Delta_- - \frac{d}{2} + 1)})) \\ &\qquad + \frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(\Delta_- - \frac{d}{2})}\ z^{\Delta_-}\ \left(\int d^d\vec{x'}\ \frac{A(\vec{x'})}{|\vec{x'} - \vec{x}|^{2\Delta_-}} + \mathcal{O}(z^2)\right).\end{aligned} \tag{F.7}$$

We are mainly interested in the situation where *both* $\Delta_\pm$ are above the unitarity bound, then $k = 0$ and A and ϕ_0 are related via the convergent position space integrals

$$A(\vec{x}) = \frac{\pi^{-\frac{d}{2}}\Gamma(\Delta_+)}{\Gamma(\Delta_+ - \frac{d}{2})} \int d^d\vec{x'} \frac{\phi_0(\vec{x'}) - \phi_0(\vec{x})}{|\vec{x'} - \vec{x}|^{2\Delta_+}}\,, \quad \phi_0(\vec{x}) \frac{\pi^{-\frac{d}{2}}\Gamma(\Delta_-)}{\Gamma(\Delta_- - \frac{d}{2})} \int d^d\vec{x'} \frac{A(\vec{x'})}{|\vec{x'} - \vec{x}|^{2\Delta_-}}\ . \tag{F.8}$$

Comparing the first formula in (F.8), containing a subtraction, with the analytic continuation from $\Delta < \frac{d}{2}$ of the corresponding formula without subtraction, we find for $\frac{d}{2} < \Delta < \frac{d}{2} + 1$

$$\int d^d\vec{x'} \frac{\phi_0(\vec{x'}) - \phi_0(\vec{x})}{|\vec{x'} - \vec{x}|^{2\Delta}} = \left(\int d^d\vec{x'} \frac{\phi_0(\vec{x'})}{|\vec{x'} - \vec{x}|^{2\Delta}}\right)_{\text{continued}}\ . \tag{F.9}$$

To check (F.9) one has to split the integral in two parts $|\vec{x'} - \vec{x}| < K$ or $> K$, use the falloff property of ϕ_0 at $|\vec{x'}| \to \infty$ and to send the arbitrary auxiliary scale K to infinity *after* the continuation. Remarkably, the singularity of the r.h.s. for $\Delta \to \frac{d}{2} - 0$ due to the short distance behavior is reproduced on the l.h.s. for $\Delta \to \frac{d}{2} + 0$ via the infrared behavior of the subtraction term.

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